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ANALOGY BETWEEN DENSITY STRATIFICATION AND ROTATION EFFECTS

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UDC 532.5.51+532.5.527

The resemblance (analogy) between the properties of rotating and density-stratified flows was first noted by Rayleigh in 1916 [1]. Since that time, a whole series of studies have been published in which this analogy is successfully employed to solve problems of wave theory and stability theory and to describe secondary regimes and turbulence. Some of these achievements are reviewed in [2, 3].

Although the successes achieved in using the analogy to obtain new results are important, our general understanding of the question is unsatisfactory. One problem is the disconnectedness of the examples with references to which the analogy has been demonstrated. The degree of proximity on the basis of which results from the two domains are considered to be analogous varies from identicalness to very distant similarity. There has been no classification of the examples of the analogy on the basis of general principles. The limits of applicability of the analogy remain unclear. The present study is an attempt to clarify these points.

From the most general standpoint, the analogy between stratification and rotation effects is a consequence of the known principle of mechanics which states that following transition to the corresponding moving frame of reference any part of the true acceleration of an object can be regarded as a "body force" field. This approach is attractive because of its simplicity and universality. However, it turns out that in all nontrivial cases it is useless owing to the velocity dependence of the "body force" field. A good example is provided by the equations of motion of a fluid written in a rotating coordinate system. Here the Coriolis force has to be taken as the "body force." Clearly, the introduction of "body forces" of this sort cannot give any basis for transposing the known results for a uniform gravitational field to a new domain.

At the same time, there are more subtle and also more productive means of explaining the analogy. At present, the only possible way of unifying the theories is mathematical. The motions of a rotating and a stratified fluid will be analogous if they are governed by equations of similar form. The degree of similarity must be such that the description of a certain class of motions in one field makes an important contribution to the solution of a related problem in the other. Given this approach, the analogy question reduces to the problem of classifying the corresponding differential equations. In general form this problem is extremely complex. The present study offers several examples illustrating the possibility of progressing along this path. Two levels of analogy, differing considerably with respect to the rigorousness of the requirements, are examined: 1) the level of similarity of the initial nonlinear equations of motion of the rotating and stratified fluids; 2) the similarity of the linearized equations of motion or their corollaries (e.g., spectral problems for linear waves and stability theory).

Comparison of the equations makes it possible to state that the properties of the motions of a rotating fluid are, generally speaking, much more complex than those of a strati-

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 58-68, May-June, 1985. Original article submitted March 22, 1984.

fied fluid. In order to obtain coincidence (or substantial similarity) of the equations of motion it is necessary to assume that the motions in the rotating fluid either possess a high degree of symmetry or are small in amplitude (or both). The analysis of the examples is directed toward establishing the limits of applicability of the analogy and studying the general qualitative properties of rotating flows.

Rigid-Body Rotation Effects

Consider the unsteady motions of an incompressible viscous fluid homogeneous with respect to density in a frame of reference rotating at the constant velocity $\Omega/2$. The equations of motion are written in the form [4]

$$D_0\mathbf{u} + \mathbf{\Omega} \times \mathbf{u} = -\nabla p^* + \nu\Delta\mathbf{u},\ \mathrm{div}\ \mathbf{u} = 0,\ D_0 \equiv \partial/\partial t + \mathbf{u}\cdot\nabla, \tag{1}$$

where $\mathbf{u}$ is the velocity vector; p^* is the modified pressure, which includes the centrifugal force; ∇ is the vector gradient; Δ is the Laplacian operator; and ν is the kinematic viscosity.

Let $\mathbf{n}$ be a unit vector representing a fixed (in the rotating frame of reference) direction and forming with vector Ω the angle θ $(0 \leqslant \theta \leqslant \pi)$. We will study the class of solutions of (1) whose velocity fields do not vary along the direction of $\mathbf{n}$.

We begin by confining our attention to the case of an ideal fluid ($\nu = 0$). We introduce the Cartesian coordinate system x, y, z, making the x axis parallel to the vector $\mathbf{n}$, i.e., $\mathbf{n} = (1, 0, 0)$. For the class of motions in question the velocity fields $\mathbf{u} = (u, v, w)$ and the pressures p^* do not depend on the x coordinate:

$$\mathbf{u} = \mathbf{u}(y, z, t),\ p^* = p^*(y, z, t). \tag{2}$$

The double angular velocity vector $\Omega = (\Omega_1, \Omega_2, \Omega_3)$.

After introducing the formal notation

$$\rho \equiv u + \Omega_2 z - \Omega_3 y,\ \mathbf{g} = (0,\ g_2,\ g_3) \equiv \mathbf{n} \times \mathbf{\Omega} = (0,\ -\Omega_3,\ \Omega_2) \tag{3}$$

we can express system of equations (1) for motions (2) in the form

$$Dv = -p_y + \rho g_2,\ Dw = -p_z + \rho g_3,\ \ D\rho = 0,\ \ v_y + w_z = 0, \tag{4}$$
$$D \equiv \partial/\partial t + v\partial/\partial y + w\partial/\partial z,$$

where $p \equiv p^* - \Omega_1\psi + (1/2)(\Omega_3 y - \Omega_2 z)^2$; ψ is the stream function for which $v = -\psi_z$, $w = \psi_y$. The indices of the independent variables everywhere denote partial derivatives.

In terms of the unknown functions ρ and ψ system (4) can be represented in the form

$$D\Delta\psi = \rho_y g_3 - \rho_z g_2,\ \ D\rho = 0,\ \ D \equiv \partial/\partial t - \psi_z\partial/\partial y + \psi_y\partial/\partial z. \tag{5}$$

For the class of steady motions the change of unknowns $v \equiv \sqrt{\rho}v_c$, $w \equiv \sqrt{\rho}w_c$ reduces system (4) to the form

$$\rho D_c v_c = -p_y + \rho g_2,\ \rho D_c w_c = -p_z + \rho g_3,$$
$$D_c\rho = 0,\ (v_c)_y + (w_c)_z = 0,\ D_c \equiv v_c\partial/\partial y + w_c\partial/\partial z. \tag{6}$$

It is immediately clear that the quantity ρ in (4)-(6) can be formally treated as the fluid density, and $\mathbf{g}$ is a uniform gravitational field. In this case system (6) coincides with the exact equations of motion of a stratified fluid, and (4), (5) with their approximate form, known as the Boussinesq approximation [5]. It should be noted that in Eq. (6) the quantities v_c and w_c are not velocity components. However, if we confine ourselves to form (6) and no-flow conditions at solid boundaries, then v_c and w_c can be formally treated as such.

The form of Eqs. (4)-(6) justifies transferring both specific results and general quantitative principles from the well-studied case of stratified flows to the motions of class (2). In order to simplify the further discussion relating to an ideal fluid, the coordinate axes y and z will be so selected that the vector Ω lies in the x, y plane. In this case $\Omega_3 = 0$ and from (3) there follows

$$\rho \equiv u + \Omega_2 z,\ \mathbf{g} = (0,\ 0,\ \Omega_2),\ \Omega_2 = \Omega\sin\theta,\ \Omega \equiv |\mathbf{\Omega}|. \tag{7}$$

The following are important examples.

1. A flow (2) of the particular form

$$u = u_0(z),\ v \equiv 0,\ w \equiv 0 \tag{8}$$

is equivalent, in terms of (4)-(6), to a state of hydrostatic equilibrium with density $\rho(z) = u_0(z) + \Omega_2 z$. The "Archimedean" stability (instability) of this equilibrium is determined by the sign of the expression

$$F = F(\theta,\ \alpha) \equiv g\rho_z = \Omega^2 \sin\theta(\alpha + \sin\theta), \tag{9}$$

where $\alpha \equiv u_{0z}/\Omega$. The domain of definition of the function $F(\theta,\ \alpha)$ is the strip $0 \leqslant \theta \leqslant \pi$, $-\infty < \alpha < +\infty$ (see Fig. 1). The equality $F = 0$ holds on the boundaries of the strip AA' ($\theta = 0$), BB' ($\theta = \pi$), and on the curve OCO' ($\alpha = -\sin\theta$). The regions $F > 0$ and $F < 0$ are located to the right and left of the curve OCO', respectively. The fact that F changes sign means that all the motions (8), at least locally, can be split into three qualitatively different groups. When $F > 0$ the "density" ρ decreases "upwards" and when $F < 0$ it increases. The equality $F = 0$ corresponds to neutral "stratification" $\rho = \text{const}$. If throughout the flow $F > 0$, then in the class of perturbations (2) the flow is stable; small perturbations take the form of internal waves. In the opposite case $F < 0$ the flow is unstable; the perturbations develop into "convective" motions.

2. Somewhat more general than (8), the flow

$$u = u_0(z),\ v = v_0(z),\ w \equiv 0 \tag{10}$$

is equivalent, in terms of (3)-(6), to a parallel flow of stratified fluid with a velocity discontinuity. The stability theory of such flows is highly developed [2, 6]. Here F again plays a key role, being the square of the Brunt–Väisälä frequency. In particular, when $F < 0$ we have a flow with a velocity discontinuity under conditions of unstable, in the Archimedean sense, "density stratification."

3. For more general fields with $u_y \neq 0$ and $\rho_y \neq 0$, where the latter means that the "density" ρ varies across the "acceleration of gravity" g, hydrostatic balance cannot exist. Therefore assigning initial data with $u_y \neq 0$, $v = w = 0$ will always lead to unsteady motion.

Triple Role of Rotation

There is a widespread qualitative notion that the general rigid-body rotation of a fluid has a stabilizing effect on any motions in it. Stabilization means a perfectly definite qualitative change in the properties of the motions on transition from $\Omega = 0$ to $\Omega \neq 0$. The following two types of changes are usually understood: The motions become wavy (i.e., internal oscillatory motions with a characteristic set of natural frequencies develop); the stability properties of the flows are "amplified."

The clearest interpretation of these ideas can be given for motions of class (2). In this case $F > 0$ corresponds to stabilization and $F < 0$ to destabilization. It is evident from Fig. 1 that even for such a narrow class of motions the effect of rotation can be qualitatively different and there is no unambiguous answer. At the same time, the notion that rotation plays a stabilizing role has a certain basis in fact. In this connection it is useful to consider the class of exact solutions of Eq. (4) expressed in terms of a single arbitrary complex function f(y):

$$\rho = \Omega_2\left[z + \mathrm{Re}\left(if e^{i\Omega_2 t}\right)\right],\ v \equiv 0,\ w = \mathrm{Re}\left(f e^{i\Omega_2 t}\right),\ p = \Omega_2^2 z^2. \tag{11}$$

These motions consist of "upward"–"downward" oscillations (in the direction of g) of the planes y = const with frequency Ω_2. The arbitrariness of f(y) means that each plane oscillates independently of the others. In terms of (4) oscillations (11) take place under the influence of the buoyancy force about the equilibrium density position. If in (11) we take

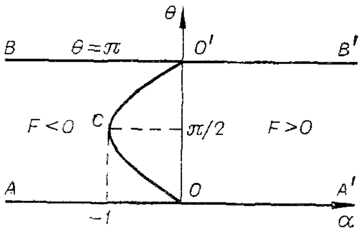

Fig. 1

$f(y) = Ae^{iky}$, we obtain the familiar Bjerknes waves [4, 7, 8], for which the constants A and k represent the complex amplitude and wave number. It should be noted that for solutions (11) the nonlinear terms of the equations of motion (1), (4) are identically equal to zero, so that (11) is a solution of both the exact equations of motion and their linearized variant. In the linear problem the Bjerknes waves play a fundamental role, since together they form a complete system of functions suitable for representing the solution of the Cauchy problem with arbitrary initial data.

From this it follows that solutions (11) give grounds for concluding that rotation has a stabilizing effect on any motions (1) of infinitely small amplitude. However, this conclusion is almost self-evident since motions with amplitudes $\alpha \to 0$ fall on the interval OO' of the $\alpha = 0$ axis (see Fig. 1) on which $F \geqslant 0$. All motions of finite amplitude (11) also fall on this same interval, but their waviness no longer provides a basis for a general conclusion.

The motions with $F = 0$, which correspond to neutral stratification with respect to the "density" ρ, proved to be especially distinctive. Without loss of generality, for these we can put $\rho = 0$ in (4). They include all motions (1) whose fields do not vary along the direction of the vector Ω. In Fig. 1 they correspond to the straight lines $\theta = 0$ and $\theta = \pi$. A noteworthy result here is the fact that the motion is not affected by the general rotation. For the same initial and boundary conditions the flow is the same in all rotating frames of reference. This result was first obtained by Taylor [9].

Thus, even within the narrow class of motions (2) the general rigid-body rotation can play three qualitatively different roles corresponding to stabilization ($F > 0$), destabilization ($F < 0$), and total lack of any effect on the motion ($F = 0$). These very different effects of rotation on different motions is the most important factor limiting the analogy between stratification and rotation effects. The limitation is a general one. In particular, it arises in connection with the problem of the stability of flows with helical and circular streamlines [10, 11].

Parallel Flows in a Rotating Layer

An example of a concrete situation, the study of which reduces to the description of motions (2), is the linear problem of the stability of a unidirectional flow in the gap between two parallel rotating planes. This problem has been studied independently [12], but many assertions can be obtained by simple transposition in the known results for parallel stratified flows [2, 6].

In the coordinate system x, y, z the positions of the planes are given by the equations $z = 0$ and $z = H$. The velocity field of the main flow has the form (8) with arbitrary function $u_0(z)$, and for (4) is the state of rest with "density" ρ (3), (7). A necessary and sufficient condition of its existence is $\mathbf{g} \times \nabla\rho = 0$. Hence there follows $\Omega_3 u_{0z} = 0$, i.e., in accordance with (7) $\Omega_3 = 0$.

Let u', v', w', p' be small perturbations of the flow satisfying Eq. (1) linearized on (8) and boundary conditions

$$w' = 0 \text{ at } \quad z = 0 \text{ and } = H. \tag{12}$$

The linear stability problem reduces to the study of the properties of the normal waves

$$[u',\ v',\ w',\ p'] = [U(z),\ V(z),\ W(z),\ P(z)]e^{i(kx+ly-\omega t)}. \tag{13}$$

Neither the main flow (8) nor the perturbations (13) vary along the direction of the vector $R\mathbf{n} = (l, -k, 0)$, $R^2 \equiv k^2 + l^2$, i.e., they belong to class (2). Therefore the study of the properties of (13) is equivalent to the stability problem for the corresponding stratified flow in the Boussinesq approximation. In order to convert (13) to the form (2) it is necessary to carry out a rotation of the coordinate system about the z axis, introducing instead of x, y the coordinates x_1, y_1 so that the x_1 axis is parallel to n and the y_1 axis parallel to the wave vector $\mathbf{k} = (k, l, 0)$. In the new coordinate system $\mathbf{k} = (0, R, 0)$. To flow (8) in terms of (4), (7) there corresponds

$$\rho = (lu_0 + Sz)/R,\ v_1 = ku_0/R,\ w_0 \equiv 0,\ \mathbf{g} = (0,\ 0,\ S/R), \tag{14}$$

where $S \equiv \mathbf{k}\cdot\Omega$; v_1 is the velocity component along y_1.

The equation for the amplitude W(z) of the plane-parallel stratified flow (14) has the known form [2, 6]

$$\tau_1^2(W_{zz} - R^2W) - R\tau_1 v_{1zz}W - R^2 g\rho_z W = 0, \tag{15}$$

where $\tau_1 \equiv -\omega + Rv_1$; $g \equiv S/R$. Substitution of (14) in (15) leads to the equation

$$\tau^2(W_{zz} - R^2W) - \tau k u_{0zz}W + R^2FW = 0. \tag{16}$$

Here we have used the notation $\tau \equiv -\omega + ku_0$, $F \equiv -\rho_z g = S(S + lu_{0z})/R$. In terms of stratified flow the quantity F is the square of the Brunt–Väisälä frequency.

The solution of the spectral problem (16), (12) for all k, l gives the answer to the question of the stability of the flow. The existence of an eigenvalue ω with $\mathrm{Im}\,\omega > 0$ signifies instability. In the particular case k = 0 we get the above-mentioned equivalence to the problem of the stability of hydrostatic equilibrium. The stability (instability) is determined by the sign of the quantity F. When $k \neq 0$, the results are less strong. The sufficient condition of stability with respect to the Richardson number (Miles–Howard theorem) [2, 6] is rewritten in the form

$$J \equiv -g\rho_z/(v_{1z})^2 = R^2F/k^2u_{0z}^2 \geqslant 1/4. \tag{17}$$

By direct transposition of the known results [2, 6] it is possible to obtain a further series of statements concerning the properties of the problem (16), (12), including the spectral limits, the stability and instability conditions, and even the results for $u_0(z)$ profiles of specific form.

However, it should be especially emphasized that despite the possibility of the literal transposition of a series of results the stability problems for stratified and rotating flows are qualitatively different. This difference finds expression in the fact that the dependence of R^2F in (16) and (17) on k and l is so strong that for any given profile $u_0(z)$ by varying k and l it is always possible to violate (17) and even cause J and F to change sign. This situation has already been examined in discussing (9) and Fig. 1. In the particular case $\Omega_1 = 0$ the sign of expression $R^2F = l^2\Omega_2(\Omega_2 + u_{0z})$ does not depend on k and l.

A very simple example with an exact solution is the problem of the stability of the linear profile $u_0(z) = \varkappa z$ with constant $\varkappa$. Equation (16) takes the form

$$\tau^2(W_{zz} - R^2W) + R^2FW = 0, \tag{18}$$

where $\tau = -\omega + k\varkappa z$; $R^2F = S(S + l\varkappa)$. For (18) we formulate the boundary-value problem in the half-space $0 \leqslant z < \infty$:

$$W(0) = 0,\ W(z) \to 0 \text{ as } z \to \infty.$$

In this case the quantity $J = S(S + l\varkappa)/(k\varkappa)^2$ does not depend on z. When $\Omega_1 \neq 0$ for any fixed $\varkappa$ by varying k and l it is possible to obtain any value of J $(-\infty < J < +\infty)$. Criterion (17) gives a sufficient condition of stability only for harmonics k and l with $J \geqslant 1/4$.

After substituting the independent variable $\xi = R[z - (\omega/k\varkappa)]$ from (18) we obtain the equation

$$W_{\xi\xi} + \left\{\left[\left(\frac{1}{4} - \nu^2\right)\Big/\xi^2\right] - 1\right\}W = 0, \tag{19}$$

where $\nu^2 \equiv (1/4) - J$. Solutions of (19) are the functions $W = \sqrt{\xi}Z_\nu(i\xi)$, where Z_ν is an arbitrary cylindrical function of index ν.

The argument ξ is varied in the complex plane along a ray departing from the point $\xi = -R\omega/k\varkappa$ and extending to infinity ($\mathrm{Re}\,\xi > 0$) parallel to the real axis. This ray corresponds to $0 \leqslant z < \infty$. The condition of damping at infinity selects from all Z_ν the Macdonald's functions, so that $W = \sqrt{\xi}K_\nu(\xi)$. It now remains to find the values of ξ causing this function to vanish. Each root will give one of the values of the combination $-R\omega/k\varkappa$. Since the point $\xi = 0$ is a branch point for $K_\nu(\xi)$, the roots must be sought on the sheet $|\arg\xi| < \pi$.

Thus the structure of the spectrum is determined by the known properties of the roots of the Macdonald's function [13]. Depending on the value of ν we get one of three qualitatively different cases:

1) ν is a purely imaginary quantity; $J > 1/4$. In accordance with (17), this is a case of a purely real spectrum. And in fact it can be shown that the Macdonald's function of purely imaginary index has a denumerable set of zeros for ξ real and positive. As $J \to 1/4$ these roots tend to the point $\xi = 0$;

2) $0 < \nu < 3/2$; $-2 < J < 1/4$. There are no roots on this interval, and no solutions of type (13) with corresponding k and l;

3) $\nu > 3/2$; $J < -2$. There are at least two complex-conjugate roots with nonzero imaginary part. When $\nu = 3/2$ these roots merge into one.

From the above it follows that the profile $u_0(z) = \varkappa z$ is unstable for any value of $\varkappa$, the instability belonging to the same type as is realized in parallel stratified flows where the density increases upwards. An exception is the previously mentioned special case $\Omega_1 = 0$. Here $J = l^2\Omega_2(\Omega_2 + \varkappa)/k^2\hat{\varkappa}^2$, and accordingly the flow is stable when $\Omega_2(\Omega_2 + \varkappa) > 0$ and unstable when $\Omega_2(\Omega_2 + \varkappa) < 0$.

Case of Nonzero Viscosity

For a viscous fluid class of motions (2) can be expanded somewhat by including in the pressure a gradient with respect to x: $-p_x^* \equiv q(t)$. Then

$$u = u(y, z, t), \; p^* = p_0(y, z, t) - q(t)x. \tag{20}$$

The corresponding generalization of Eq. (4) is written in the form

$$\begin{gathered} D_1 v = -p_y + \rho g_2, \; D_1 w = -p_z + \rho g_3, \\ D_1 \rho = q, \; v_y + w_z = 0, \; D_1 \equiv D - \nu(\partial^2/\partial y^2 + \partial^2/\partial z^2). \end{gathered} \tag{21}$$

The form (21) coincides with the equations of a stratified fluid only when $q = 0$. This case corresponds either to flows associated with the motion of boundaries or unsteady flows. When $q \neq 0$ fields (20) include, in particular, the interesting class of confined flows in rotating cylindrical tubes with the generator of the cylinder parallel to the x axis. There are no constraints on the shape of the tube cross section or on the angle θ between the vector Ω and the generator. At the tube walls the no-slip condition leads to the requirements

$$\rho = \Omega_2 z - \Omega_3 y, \; v = 0, \; w = 0. \tag{22}$$

After this the description of the motion of a viscous fluid in a rotating tube reduces to problem (21), (22) for two-dimensional density-stratified flows. The latter problem is very unusual since it contains sources of "density" ρ and at the boundaries the values of ρ are given. A more natural interpretation of (21), (22) is obtained by replacing "density" ρ by "temperature" T (i.e., by introducing the equation of state $\rho = T$). As a result, at the boundaries the "temperature" distribution is given, and the equations $D_1 T = q$ and $v_y + w_z = 0$ correspond to the presence of "heat sources" that change the "temperature" and "density" without expansion (or compression) of the fluid.

For problem (21), (22) the following statements are obvious, both mathematically and physically:

1) When $q = 0$ the unique stationary solution is $\rho = \Omega_2 z - \Omega_3 y$, $v = w = 0$ (rigid-body rotation of the fluid);

2) when $q = \text{const} \neq 0$ there are no stationary solutions with $v = w = 0$ (corresponding to purely longitudinal flow along the tube). The flows with transverse circulation, observed in [14, 15], are obtained. The physical cause of this circulation, in terms of (21), is that the "heavy" fluid sinks while the "light" fluid rises.

In the theory of stratified fluids and in the theory of convection a formulation of type (21), (22) is artificial and has not been considered. Accordingly, generally speaking, the presence of viscosity imposes a limit on the useful application of the analogy. At the same time, for stability problems there is no such limitation. In fact, linearized equations (21), (22) coincide with the corresponding linear problem of the stability of a viscous stratified fluid with density diffusion.

Ekman Flows

A simple variant of problem (21), (22) is the confined flow in the gap between two parallel planes. The Cartesian coordinate system x, y, z is introduced so that $z = 0$ and $z = H$ correspond to the position of the planes. The applied pressure gradient is constant and directed along the x axis: $p_x = -q = \text{const}$. In the plane y, z the flow region is a strip $0 < z < H$. The vector Ω_n is the projection of Ω on the plane y, z. The "gravitational field" $g = (0, -\Omega_3, \Omega_2)$ is perpendicular to Ω_n. The corresponding stationary solution (21), (22) is written in explicit form.

Since the field g has a nonzero component tangential to the planes, it is natural to expect a flow with slippage:

$$v = v(z),\ w \equiv 0,\ \rho = -\Omega_3 y + \Omega_2 z + u(z), \tag{23}$$

which represents the "downward" motion of the fluid under the influence of the "force of gravity." In the transverse direction (along the z axis) the "force of gravity" is balanced by the pressure gradient. Substitution of (23) in (21), (22) leads to a boundary value problem for a system of ordinary differential equations:

$$\nu u_{zz} + \Omega_3 v + q = 0,\ \nu v_{zz} + \Omega_3 u = 0,$$

$$u = v = 0 \quad \text{at} \quad z = 0,\ z = H.$$

The solution of this problem is rather clumsy and will not be written out. When $H \to \infty$ it takes the simpler form

$$u = \frac{q}{\Omega_3} e^{-kz} \sin kz, \quad v = -\frac{q}{\Omega_3}(1 - e^{-kz}\cos kz). \tag{24}$$

Here $k \equiv \sqrt{\Omega_3/2\nu}$. Relations (23), (24) show that the velocity component u, directed along the pressure gradient, is rapidly damped with distance from the wall, whereas the transverse component v increases. Consequently, the main flow (in a sufficiently wide channel) is directed across the applied pressure gradient.

Thus, formulating Eq. (1) in terms of (21), (22) makes it possible to obtain a clear idea of the mechanics of Ekman flow. This flow is equivalent to the motion of a stratified fluid in the gap between two inclined (to the gravitational field) planes. At the same time, it is possible to formulate interesting hypotheses about the stability properties of such a flow. In particular, the sign of the quantity $\rho_z g_3$ indicates an increase or decrease in "density" along the component of g in the transverse direction relative to the plates. For (24) we obtain

$$\rho_z g_3 = \Omega_2(\Omega_2 - u_z) = \Omega_2^2\left[1 - \frac{kq}{\Omega_2\Omega_3} e^{-kz}(\cos kz - \sin kz)\right]. \tag{25}$$

For $q/\Omega_2\Omega_3 > 0$ (25) has its lowest value at $z = 0$. Accordingly, for sufficiently large pressure drops ($q > \Omega_2\Omega_3/k \equiv \Omega_2\sqrt{2\nu\Omega_3}$) we get a layer of fluid in which the "density" increases "upwards" and "convective" instability is possible.

Nonrigid-Body Rotation Effects

Above, in studying rigid-body rotation effects we made an artificial distinction between the rotation of the fluid as a whole and its internal motions, and examined the influence of the former motion on the latter. This is a very special approach. The obvious next step is to study nonrigid-body rotation effects, without dividing the motion into parts. As an example of this more general approach let us consider the class of motions with helical symmetry.

We introduce the cylindrical coordinate system φ, r, z in which the components of the velocity vector are u, v, and w, respectively. The initial system of equations of motion has the form

$$\begin{gathered} Du + uv/r = -p_\varphi/r\rho_1,\ Dv - u^2/r = -p_r/\rho_1, \\ Dw = -p_z/\rho_1,\ v_r + v/r + u_\varphi/r + w_z = 0,\ D\rho_1 = 0, \\ D \equiv \partial/\partial t + v\partial/\partial r + (u/r)\partial/\partial\varphi + w\partial/\partial z, \end{gathered} \tag{26}$$

where ρ_1 is the density of the fluid. To begin with, let $\rho_1 \equiv 1$. These motions with helical symmetry are characterized by the fact that the corresponding solutions of (26) are functions of the three independent variables t, r, and $\mu \equiv a\varphi - bz$. Here a is any natural number, and b any real number.

Using the notation $\rho \equiv (bru + aw)^2$, $\lambda \equiv au - brw$, $R \equiv a^2 + b^2r^2$, $g \equiv b^2r/R^2$, $K \equiv 2ab/R^2$, we can reduce Eq. (26) for motions with helical symmetry to the form

$$\begin{gathered} D_\mu(r\lambda/R) + K\sqrt{\rho}rv = -p_\mu,\ D_\mu v - K\sqrt{\rho}\lambda - (a\lambda/R)^2/r = -p_r + \rho g, \\ D_\mu\rho = 0,\ v_r + v/r + \lambda_\mu/r = 0,\ D_\mu \equiv \partial/\partial t + v\partial/\partial r + (\lambda/r)\partial/\partial\mu. \end{gathered} \tag{27}$$

This system is similar to the equations of motion of a stratified fluid. The similarity consists in the existence of a dynamical analogue of the density ρ, preserved in each fluid particle, and in the form of the force ρg on the right side of the second equation. At the same time, there are important differences, in particular the presence of the terms with coefficient K. The structure of these terms recalls the Coriolis force. The following question immediately arises: Can system (27) be regarded as analogous to the equations of motion of a stratified fluid? The answer reduces to determining the degree of similarity that can be identified as an analogy.

In two particular cases the answer is obvious. Thus, $a = 0$ corresponds to the class of symmetrical rotational motions for which (27) reduces to the form

$$Dv = -p_r + \rho g,\ Dw = -p_z,\ D\rho = 0,\ v_r + v/r + w_z = 0,\ D \equiv \partial/\partial t + v\partial/\partial r + w\partial/\partial z, \tag{28}$$

where $g = 1/r^3$; $\rho = (ru)^2$; without loss of generality b has been taken equal to -1. System (28) coincides with the equations of a stratified fluid in the Boussinesq approximation with the body force field directed along the radius. Therefore, the conclusions concerning the degree of similarity of the stratification and rotation effects for symmetrical rotational motions practically repeat the results for (4)-(6). The analogue for the quantity F of (9) in (28) is

$$F_1 \equiv g\rho_r = \frac{1}{r^3}\frac{d}{dr}(ru)^2 = 2u\Omega/r,$$

where Ω is the axial component of the vorticity vector $\Omega \equiv u_r + u/r$. The condition $F_1 > 0$, signifying the Archimedean stability of the state of rest ($v \equiv 0$, $w \equiv 0$), is widely known as the Rayleigh stability criterion [1, 4, 7]. When $a = 1$, $b = 0$ the motions described by (27) are plane and do not depend on the z coordinate. The term with the body force ρg is equal to zero. These motions were previously examined as the case $F = 0$ in (4), (9).

In the general case of arbitrary a and b it is useful to write out the expressions for the vorticity. Let η, ω_1, and ω_2 be the vorticity components in the directions of the radius and μ = const and in the direction perpendicular to them. The following representations hold:

$$\omega_1 = \sqrt{R}(\omega_0 + K\sigma),\ \omega_2 = -\sigma_r/\sqrt{R},\ \eta = \sigma_\mu/r,\quad \omega_0 \equiv (1/r)[(r\lambda/R)_r - v_\mu],\ \sigma \equiv \sqrt{\rho}. \tag{29}$$

From the first two of Eqs. (27), after eliminating the pressure, we obtain the relation

$$D_\mu\omega_0 + \left(v_0\frac{\partial}{\partial r} + \frac{\lambda}{r}\frac{\partial}{\partial\mu}\right)K\sqrt{\rho} + \frac{b^2}{R^2}\rho_\mu = 0, \tag{30}$$

which is an analogue of (5).

Particularly interesting is the question of the qualitative role of the quantity ρ in (27), (30). Here the investigation of the general case is very complicated. At the same time, when $\rho \neq$ const, the existence of wave motions (real natural frequencies) in the linear approximation is widely known. As for the properties of the motions in the exact formulation (27), it can be seen that at least flows with ρ = const are very similar to nonrotating flows of a homogeneous fluid. Here it should be noted that by virtue of the equation $D_\mu\rho = 0$ flows with ρ = const form an independent class. Since the quantity $\sigma \equiv \sqrt{\rho}$ is determined correct to a constant, the case ρ = const reduces to $\rho = 0$ after going over to the corresponding frame of reference moving with constant velocity along the z axis. Equations (27), (30) show that flows with $\sigma = 0$ are equivalent to the previously examined flows with $b = 0$. In particular, when $\rho = 0$, Eq. (30) degenerates into a condition of the type of conservation of vorticity in each fluid particle. Then, in accordance with (29), flows with $\rho = 0$ have a single nonzero vorticity component ω_1.

The conclusions reached in this section can be formulated as two propositions. Firstly, the presence of helical symmetry (as distinct from motions (2)) does not allow the description of the motions of a rotating fluid to be reduced to the description of the motions of a stratified fluid. Nonetheless, Eqs. (27), (30) for motions with helical symmetry are very similar to the equations of a stratified fluid. Secondly, the quantity ρ in (27), (30) plays the role of density in the broader sense: When ρ = const there are not any internal wave motions, whereas when $\rho \neq$ const, generally speaking, there are.

Relaxation of the Requirements

Now, in (26) let the true density $\rho_1 \neq$ const. The exact solution of (26) is the flow with helical streamlines

$$u = u_0(r),\ v \equiv 0,\ w = w_0(r),\ \rho_1 = \rho_0(r). \tag{31}$$

We will consider small-amplitude perturbations satisfying system of equations (26) linearized on (31). The problem reduces to the study of perturbations in the form of normal waves:

$$v'(r,\ \varphi,\ z,\ t) = \mathrm{Re}\ V(r)e^{i(kz+m\varphi-\omega t)}. \tag{32}$$

Here V(r) is the complex amplitude function. After substitution of representations of the type (32) for the perturbations u', v', w', p', and ρ_1' in (26) and a series of manipulations, it is possible to obtain an equation for the single function $\psi(r) \equiv rV(r)$ [10]:

$$\tau^2\left[\psi_{rr} + \left(\frac{1}{r} - \beta - \beta_1\right)\psi_r - R\psi\right] + RF_2\psi + \left\{\frac{m}{r}\left[-\Omega_r + (\beta+\beta_1)\Omega\right] - k\left[\frac{1}{r}(rw_{0r})_r - (\beta+\beta_1)\,w_{0r}\right]\right\}\tau\psi = 0, \tag{33}$$

where $\tau \equiv -\omega + m(u_0/r) + kw_0$; $R \equiv k^2 + (m^2/r^2)$; $\rho \equiv (m^2 + k^2r^2)^{-1}$; $\beta \equiv -\rho_r/\rho$; $\beta_1 \equiv -\rho_{1r}/\rho_1$; $\Omega \equiv u_{0r} + u_0/r$; $F_2 \equiv \beta G + \beta_1 G_1$; $G \equiv u_0(\Omega - mw_{0r}/kr)$; $G_1 \equiv u_0^2/r$. A remarkable property of Eq. (33) is the fact that the quantities associated with stratification and rotation enter into the coefficients symmetrically. The formally introduced quantities ρ and G enter into the equation along with the density ρ_1 and the centrifugal gravitational field G_1. The quantity $\sqrt{F_2}$ is a generalization of the buoyancy (Brunt–Väisälä) frequency [2, 6].

The no-flow boundary conditions for flow between coaxial circular cylinders of radii R_1 and R_2 ($R_1 < R_2$) have the form

$$\psi(R_1) = \psi(R_2) = 0. \tag{34}$$

Problem (33), (34) is a spectral problem for the determination of the eigenvalues ω and the eigenfunctions $\psi(r)$. The existence of an eigenvalue with $\mathrm{Im}\,\omega > 0$ signifies instability.

For problem (33), (34) the statement generalizing the known Miles–Howard theorem [2, 6] holds. A sufficient condition of stability (realness of the spectrum ω) is the satisfaction of the inequality [10, 16]

$$J \equiv \left(k^2 + \frac{m^2}{r^2}\right)F_2\Big/\left[kw_{0r} + m\left(\frac{u_0}{r}\right)_r\right]^2 \geqslant 1/4,$$

where J is a generalization of the Richardson number.

It should be stressed that as distinct from ρ_1 and G_1 their analogues ρ and G proved to depend on the form of the perturbation. Therefore, the analogy exists separately for each fixed pair of wave numbers k, m. The situation resembles that already discussed in connection with (14).

The strongest result is obtained for symmetrical rotational perturbations (m = 0). In this case from (33) there follows the equivalence of the influence on the perturbations of stratification and rotation, which implies the possibility of replacing rotation by density stratification (and vice versa) without changing the form of the equation. This statement is a particular case of the reduction of the complete nonlinear equations of motion to the form (28). In the general case there is no such equivalence, and on the basis of (33) it is possible to speak only of the similar influence of stratification and rotation on the perturbations, the dependence of G on k and m being so important that for different perturbations the same fields $u_0(r)$ and $w_0(r)$ correspond to different signs of G (cf. (9), (14), and (17)).

Both flows (31) and perturbations (32) belong to the class of motions with helical symmetry (see previous section). However, there is an important difference between the forms of demonstration of the analogy (27) and (33). In particular, the quantities ρ and G in (33) do not coincide with ρ and g in (27). The only aim pursued in introducing ρ and G in (33) was to achieve the maximum symmetry in the form of Eq. (33). In rewriting (33) in terms of notation (27) such strict symmetry cannot be obtained. At the same time it is clear that the forms of analogy (27) and (33) are qualitatively similar. Thus, the analogues of the buoyancy frequencies $\rho_r g$ in (27) and βG in (33) are alike. In particular, $\rho \equiv 0$ in (27) is equivalent to $G \equiv 0$ in (33).

Thus, approach (33) with its relaxed requirements makes it possible to find more fruitful variants of the analogy. A whole series of concrete results obtained on the basis of (33) are presented in [10, 11, 17, 18].

In conclusion, it should be noted that all the motions of a rotating fluid considered above possess a high degree of symmetry. The extremely interesting question of the extent to which the analogy applies to rotational motions of more general form remains open. An attempt to answer this question is made in [19], which includes a generalization of the so-called Taylor–Proudman theorem. This is closely analogous to the well-known blocking effect in a stratified fluid [2]. Unfortunately, this generalization was obtained only for narrow classes of rotational flows.

The author wishes to thank L. V. Ovsyannikov for useful discussions.

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